Lecture 15

Plan

1) Finish Min T-odd cut ✓
(see Lec14 notes]

2) Matroids.

- Pset 4 deadline extended to Mon Apr 26

- No OH upcoming Monday evening

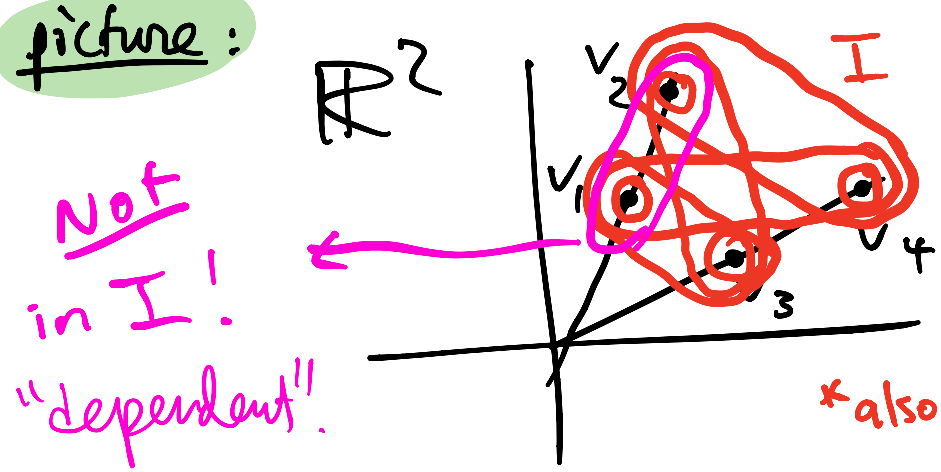
Matroids

"Tractable" set systems.

Eg. Given vectors $v_1, \ldots\ldots, v_m \in \mathbb{R}^n$, consider set system $I \subseteq 2^{[m]}$:

$$I = \left\{ S \subseteq [m] : \{v_i : i \in S\} \text{ linearly independent} \right\}$$

picture:



*also $\emptyset \in I$.

Properties of $\mathcal{I}$:

(P1) "Downward closed"

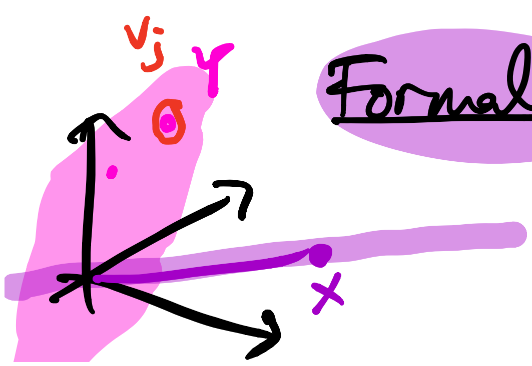
If $X \subseteq Y$, $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

(P2) "Exchange property"

If $X \in \mathcal{I}$ and $Y \in \mathcal{I}$ and

$$|Y| > |X|,$$

then $\exists$ elt. in Y you can add to X while maintaining indep. of X.



Formally: $\exists e \in Y \setminus X$ s.t. $X \cup \{e\} \in \mathcal{I}$.

$X + e$

Pf of P2: $|X| = \dim \operatorname{span}\{v_i : i \in X\}$
$|Y| = \dim \operatorname{span}\{v_i : i \in Y\}$.

$\Rightarrow \operatorname{span}\{v_i : i \in X\} \not\supseteq \operatorname{span}\{v_i : i \in Y\}$.

$\Rightarrow \exists j \in Y$ s.t. $v_j \notin \operatorname{span}\{v_i : i \in X\}$.

$\Rightarrow X + j \in I$. $\square$

P1, P2 capture combinatorial structure of I.

for matroids: take P1, P2 as axioms.

$I \subseteq 2^E$, X is maximal in I if $\exists$ no $Y \in I$ s.t. $X \subsetneq Y$.

Def (Matroid) A matroid M is a pair (E, I) where

- $E = E(M)$ finite set called ground set of M.
- $I = I(M) \subseteq 2^E$ called independent sets.

- $\mathcal{I}$ satisfies P1 & P2.

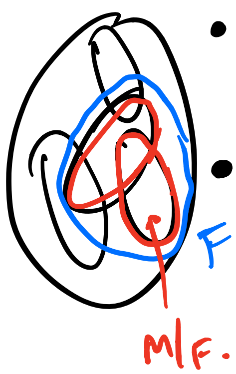
maximum

Remarks

- P2 $\Rightarrow$ all maximal (by inclusion) indep sets have same size. (else could increase by P2).

- Maximal independent set called a **base** of M.
- **dependent** := not independent
- for $F \subseteq E$, the **restriction** $M|_F$

$M|_F = \{S \in \mathcal{I} : S \subseteq F\}$ is another matroid.



Examples

AKA "representable"

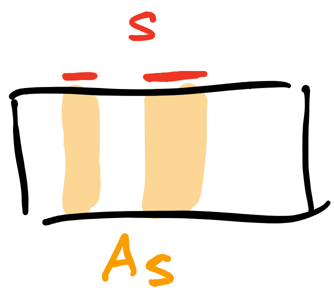
- **Linear matroid**: example from beginning.

▷ **equiv def**: $A \in \mathbb{R}^{n \times m}$ matrix,

$\mathcal{I} = \{$ subsets S of cols. of A
s.t. submatrix A_S
has rank $A_S = |S|\}$.

$E = \{$set of columns of $A\}$. $|E| = m$

e.g.

A :


write $M = M_A$
if M comes
from A.

▷ This makes sense for $A \in \mathbb{F}^{n \times m}$
for any field $\mathbb{F}$.

▷ bases of M_A: subset S s.t.
cols. of A_S are a basis for $\mathbb{R}^n$.

- "boring" example:
uniform matroid: $U_{n,k} = (E, \mathcal{I})$
where $|E| = n$

$U_{n,2}$
complete
graph on

$\mathcal{I} = \{$all subsets of E of size $\le k\}$
$= \{S \subseteq E : |S| \le k\}$.

n vertices.

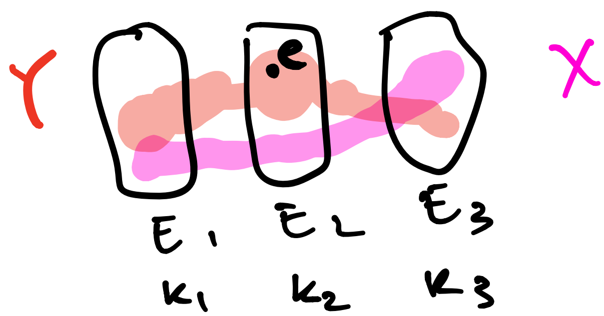
the free matroid is $U_{n,n} = 2^E$.

- partition matroid: $M = (E, \mathcal{I})$ where E is disjt. union $E_1 \dot\cup \dots \dot\cup E_\ell$

$$\mathcal{I} = \{X \subseteq E : |X \cap E_i| \le k_i\}$$

for fixed $k_1 \dots . k_\ell$ (parameters).

e.g.



Check P2:

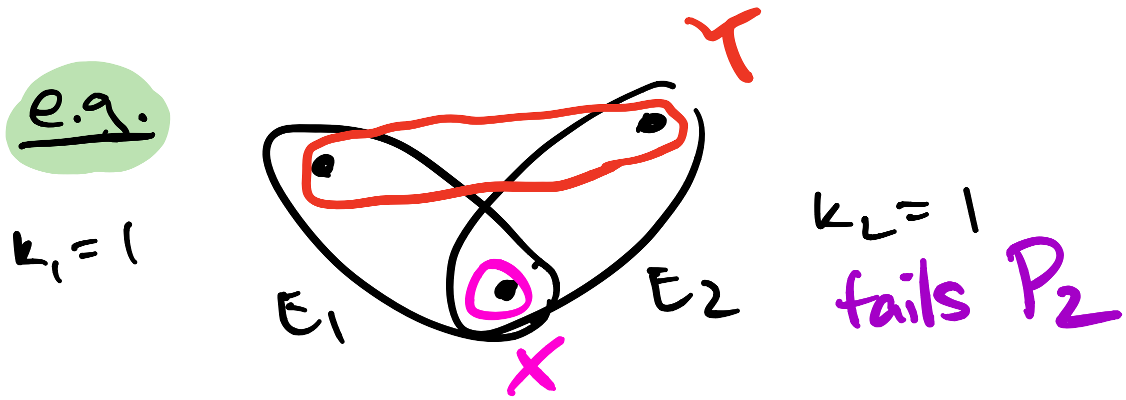
- Let $|X| < |Y|$, $X, Y \in \mathcal{I}$.
- $\exists\, i$ s.t. $k_i \geq |Y \cap E_i| > |X \cap E_i|$

• ∃ for any $e \in Y \cap E_i \setminus X \cap E_i$

$X + e$ independent.

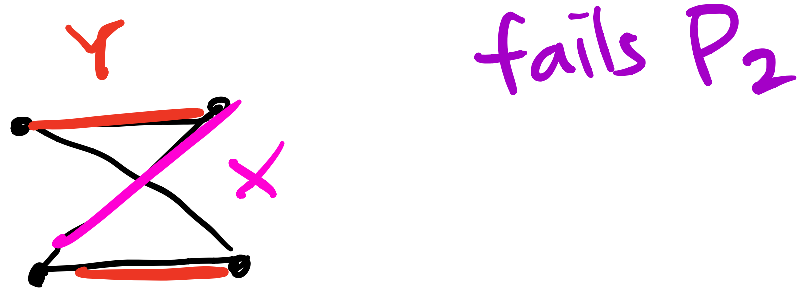
Remark: if E_i not disjoint:

e.g.



• **Another Nonexample:** set of matchings in a graph.

e.g.



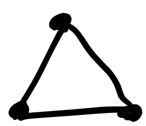
- <u>**graphic matroids**</u>:

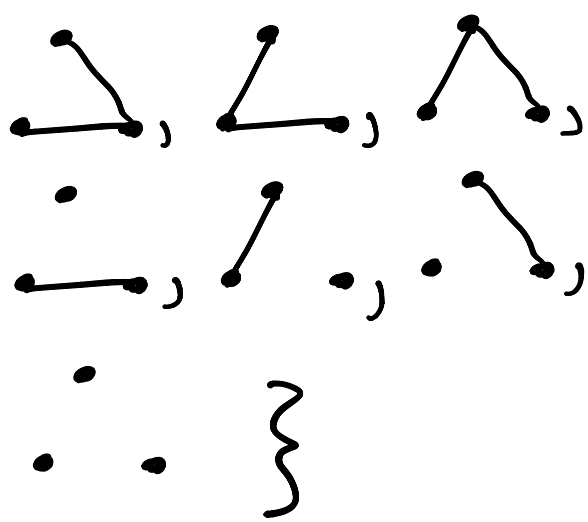
  Given graph $G=(V,E)$, undirected

  let $M(G) = (E, I)$ where

  $I = \{$ <u>forests</u> in G $\}$

  $= \{$ acyclic subgraphs of G $\}$.

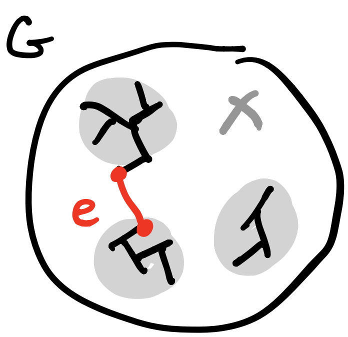
<u>e.g.</u> $G =$ 

$I = \{$  $\}$

Checking P2:

- F forest $\Rightarrow |V| - \overset{\#\text{ edges}}{|F|} =$ # connected components (c.c.'s)
- X, Y forests, $|X| < |Y| \Rightarrow Y$ has fewer c.c.'s.

$\Rightarrow$ some edge e of Y connects two c.c.'s of X

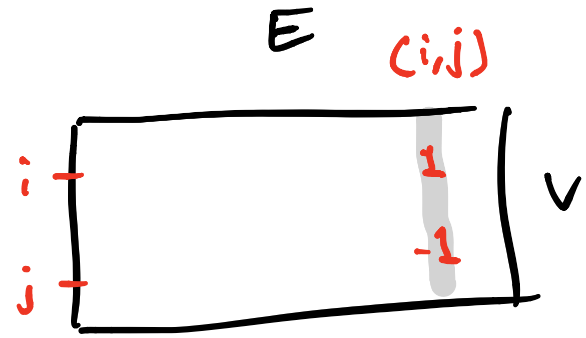


$\Rightarrow X + e$ larger forest.

▷ <u>bases</u>: the spanning trees.
(all have $n-1$ edges).

▷ <u>graphic $\Rightarrow$ linear</u>:

Pf: $M(G) = M_A$ where A directed vertex-edge incidence matrix (direct arbitrarily).

$A =$ 

Ex. Check: subset of cols. is lin. indep $\iff$ subgraph contains no cycle. $\square$

▷ Graphic $\Rightarrow$ regular:

Say M regular if M is linear over every field $\mathbb{F}$.

-1 in A $\leftarrow$ additive inverse of 1 in $\mathbb{F}$.

Note: A above is T.U.

Fact: matroid M regular $\iff$ $M = M_A$ (over $\mathbb{R}$) for T.U. matrix A.

Circuits

- **circuit:** = minimal dependent set.
  by inclusion.
  (i.e. C circuit $\Rightarrow C-e$ independent).

e.g. ▷ in graphic matroid: circuits are the cycles.

▷ in partition matroid, circuits are just subsets $C \subseteq E_i$ with $|C| = k+1$.

e.g.

Note:

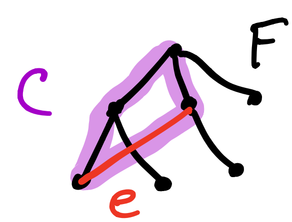
C circuit $\Rightarrow C-e$ independent

There's exactly one way to do the reverse:

Theorem (unique circuit property)

- Let $M=(E,\mathcal{I})$ matroid.
- Let $S\in\mathcal{I}$, $e\in E$ s.t. $S+e\notin\mathcal{I}$
- Then: $\exists$ a unique circuit $C\subseteq S+e$.

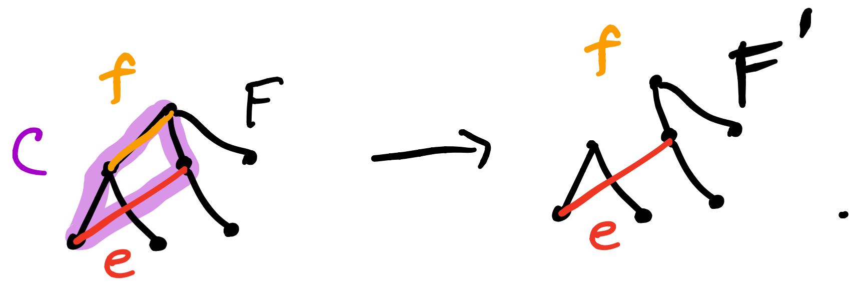
e.g. If F is a forest, $F+e$ isn't:



Remark: uniqueness shows

how to make more independent sets: Let $C \subseteq S+e$ circuit, $f \in C \setminus e$ then $S+e-f \in \mathcal{I}$.

e.g.



Pf: Else, $S+e-f$ contains circuit $C' \neq C$ (and hence $S+e$).

Proof of UCP:

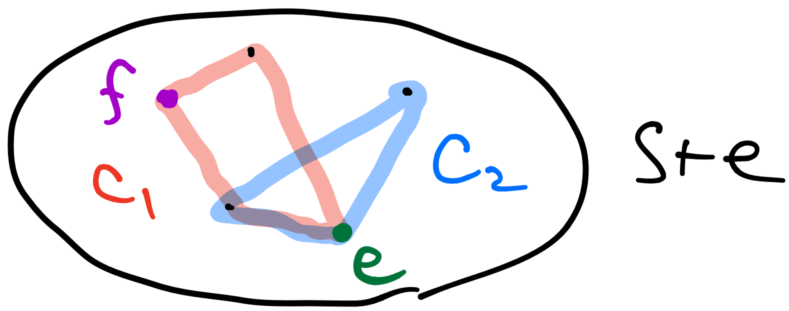
- suppose $S+e$ contains distinct circuits $C_1 \neq C_2$.

was typo in class :(

- Minimality $\Rightarrow$ $C_1 \not\subseteq C_2$ $\Rightarrow$ $\exists f \in C_1 \setminus C_2$.

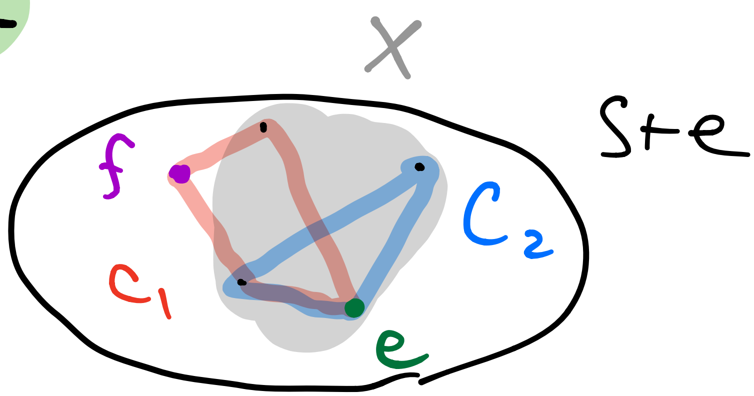
Note: $C_2 \subseteq S+e-f$.

▷ We'll show $S+e-f \in \mathcal{I}$; contradicts $C_2 \subseteq S+e-f$. (bc C_2 dependent, P1).



- $C_1 - f$ independent (by minimality) $\Rightarrow$ $C_1 - f$ to maximal indep. X subject to $X \subseteq S+e$.

e.g.



or $M|_{S+e}$ is matroid,

- Both S, X maximal independent within $S+e$ $\Rightarrow$ $|X| = |S|$ by P2.

- $e \in X \Rightarrow$ because $e \in C_1 - f \subseteq X$
  $\Rightarrow X = S + e - f$.

- $\Rightarrow S + e - f$ indep, contradiction. $\square$

$M = (E, I)$ e.g. could have $E =$ edge set of G
$I =$ forests in G.
(review P1, P2 from earlier).

Matroid optimization

- Given cost function $c : E \to \mathbb{R}$,
  want indep. set S of max. cost
$$c(S) = \sum_{e \in S} c(e).$$

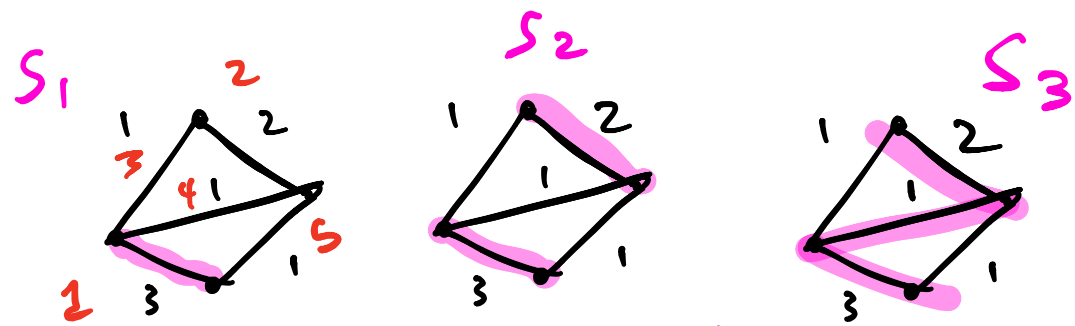
- This problem is tractable,
  one reason matroids are important.

- if some $c(e)<0$: can restrict to $M|_{E-e}$. (removing e from $S \in \mathcal{I}$ increases cost).

- if $c \geq 0$: need only optimize over <u>bases</u>.

<u>e.g.</u> for graphic matroids: on connected graphs this is the maximum spanning tree problem (MST).

<u>Recall</u>: M.S.T. has simple <u>greedy</u> algorithm: keep adding largest element that doesn't create a cycle.



Kruskal's algorithm

- **Fact**: greedy alg works for any matroid.
- Actually, for all k: greedy outputs indep set of size k of max cost. S_k

Algorithm

Let $|E|=m$.

▷ Sort E by cost: $c(e_1)\geq c(e_2)\ldots\geq c(e_m)$

▷ $S_0 := \emptyset$, $k=0$

▷ For $j = 1$ to m:

▷ if $S_k + e_j \in \mathcal{I}$ then:

▷ $S_{k+1} := S_k + e_j$

▷ $k \leftarrow k+1$.

▷ Output $S_1 \ldots S_k$.

Thm: For any matroid $M = (E, \mathcal{I})$, above alg. finds indep. set S_k such that

$$c(S_k) = \max_{\substack{|S| = k \\ S \in \mathcal{I}}} c(S).$$

Proof: Suppose not.

- Let $S_k = \{s_1, \dots, s_k\}$ with $c(s_1) \geq \dots \geq c(s_k)$.

- Suppose $T_k = \{t_1, \dots, t_k\}$
  $c(t_1) \geq c(t_2) \dots \geq c(t_k)$
  s.t. $c(T_k) > c(S_k)$.

- Let $p :=$ first index where
  $c(t_p) > c(s_p)$.

- Let $A = \{t_1, \dots, t_p\}$
  $B = S_{p-1} = \{s_1, \dots, s_{p-1}\}$.

- $|A| > |B| \Rightarrow \exists\, t_i \in A \setminus B$ s.t. $B + t_i \in \mathcal{I}$ (by P2).

- But $c(t_i) \geq c(t_p) > c(s_p)$
  $\Rightarrow c(t_i) > c(s_p)$
  $\Rightarrow t_i$ should have been added to S_{p-1} instead of s_p. ※. $\square$

To get global ~~min~~ max-cost independent set:

In greedy alg,

Replace for $j = 1 \ldots m$

for $j = 1 \ldots q$

~~[illegible]~~ e_q is last nonnegative element.